Cohomology ring : Solution Sheet 5

◊ **Exercice 1. Functoriality.** Consider the inclusion $i\colon C_2 \hookrightarrow C_4$ and the projection $p\colon C_4 \to C_2$. Let $F_\bullet$ be the free periodic $\mathbb{Z}C_2$-resolution of $\mathbb{Z}$ and $G_\bullet$ the free periodic $\mathbb{Z}C_4$-resolution.

1. Extend the identity on $\mathbb{Z}$ to a map $\tau\colon F_\bullet \to G_\bullet$ of $\mathbb{Z}C_2$-chain complexes.
2. Compute the induced map $H_n(C_2;\mathbb{Z}) \to H_n(C_4;\mathbb{Z})$ for all $n \geq 0$.
3. Compute the induced map $H_n(C_2;\mathbb{F}_2) \to H_n(C_4;\mathbb{F}_2)$ for all $n \geq 0$.
4. Extend the identity on $\mathbb{Z}$ to a map $\tau\colon G_\bullet \to F_\bullet$ of $\mathbb{Z}C_4$-chain complexes.
5. Compute the induced map $H_n(C_4;\mathbb{Z}) \to H_n(C_2;\mathbb{Z})$ for all $n \geq 0$.
6. Compute the induced map $H_n(C_4;\mathbb{F}_2) \to H_n(C_2;\mathbb{F}_2)$ for all $n \geq 0$.

Solution 1. (by Eliot and Damien)

1. First observe that we can see $\mathbb{Z}C_4$ as a $\mathbb{Z}C_2$-module thanks to $i : \mathbb{Z}C_2 \to \mathbb{Z}C_4$ so that $t \cdot x = t^2 x$ for $t \in \mathbb{Z}C_2$ and $x \in \mathbb{Z}C_4$. Let's extend the identity on $\mathbb{Z}$ to a map $f : F_\bullet \to G_\bullet$, so we need to find $\mathbb{Z}C_2$ homomorphisms f_n for all $n \in \mathbb{N}$ such that the following diagram commutes :

$$\begin{tikzcd}
... \arrow[r] & \mathbb{Z}C_2 \arrow[r, "t-1"] \arrow[d, "f_3"] & \mathbb{Z}C_2 \arrow[r, "t+1"] \arrow[d, "f_2"] & \mathbb{Z}C_2 \arrow[r, "t-1"] \arrow[d, "f_1"] & \mathbb{Z}C_2 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[d, "id"] \\
... \arrow[r] & \mathbb{Z}C_4 \arrow[r, "t-1"] & \mathbb{Z}C_4 \arrow[r, "t^3+t^2+t+1"] & \mathbb{Z}C_4 \arrow[r, "t-1"] & \mathbb{Z}C_4 \arrow[r, "\epsilon"] & \mathbb{Z}
\end{tikzcd}$$

Since the f_n's are $\mathbb{Z}C_2$ homomorphisms, they're totally determined by the image of 1. Observe that we can take $f_0 = i$ and then we need that the following square commutes :

$$\begin{tikzcd}
\mathbb{Z}C_2 \arrow[r, "t-1"] \arrow[d, "f_1"] & \mathbb{Z}C_2 \arrow[d, "i"] \\
\mathbb{Z}C_4 \arrow[r, "t-1"] & \mathbb{Z}C_4
\end{tikzcd}$$

Thus we get :

$$f_1(1)(t-1) = t^2 - 1$$

Then, $f_1(1) = t+1$ works. We'll denote $f_1 = j$, observe $j(t) = t^3 + t^2$. Now we need that the following square commutes :

$$\begin{tikzcd}
\mathbb{Z}C_2 \arrow[r, "t+1"] \arrow[d, "f_2"] & \mathbb{Z}C_2 \arrow[d, "j"] \\
\mathbb{Z}C_4 \arrow[r, "t^3+t^2+t+1"] & \mathbb{Z}C_4
\end{tikzcd}$$

Thus we get :

$$f_2(1)(t^3 + t^2 + t + 1) = t^3 + t^2 + t + 1$$

Then, $f_2(1) = 1$ works. We write $f_2 = i$. Using the periodicity of both complexes, we deduce that $f_{2n} = i$ and $f_{2n+1} = j$ for $n \in \mathbb{Z}$:

$$\begin{tikzcd}
... \arrow[r] & \mathbb{Z}C_2 \arrow[r, "t-1"] \arrow[d, "j"] & \mathbb{Z}C_2 \arrow[r, "t+1"] \arrow[d, "i"] & \mathbb{Z}C_2 \arrow[r, "t-1"] \arrow[d, "j"] & \mathbb{Z}C_2 \arrow[r, "\epsilon"] \arrow[d, "i"] & \mathbb{Z} \arrow[d, "id"] \\
... \arrow[r] & \mathbb{Z}C_4 \arrow[r, "t-1"] & \mathbb{Z}C_4 \arrow[r, "t^3+t^2+t+1"] & \mathbb{Z}C_4 \arrow[r, "t-1"] & \mathbb{Z}C_4 \arrow[r, "\epsilon"] & \mathbb{Z}
\end{tikzcd}$$

2. Let's recall that

$$H_n(C_2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n=2k \\ \mathbb{F}_2 & \text{if } n=2k-1 \end{cases}$$

for $k \in \mathbb{N}_{\geq 1}$. And

$$H_n(C_4;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n=2k \\ \mathbb{F}_4 & \text{if } n=2k-1 \end{cases}$$

for $k \in \mathbb{N}_{\geq 1}$. Now we can apply to our previous diagram the functor $\mathbb{Z} \otimes_{\mathbb{Z}C_2} \_$ and then change the base ring of the bottom row's tensor products $\mathbb{Z} \otimes_{\mathbb{Z}C_2} G_\bullet \to \mathbb{Z} \otimes_{\mathbb{Z}C_4} G_\bullet$ to get chain complexes, and then the maps i and j become :

$$\begin{tikzcd}
... \arrow[r] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "2"] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "1"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "2"] & \mathbb{Z} \arrow[d, "1"] \\
... \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "4"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}
\end{tikzcd}$$

And thus in the homology groups it becomes :

$$H_0(f;\mathbb{Z}) = id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$$
$$H_{2k}(f;\mathbb{Z}) = 0 : 0 \to 0$$
$$H_{2k-1}(f;\mathbb{Z}) = \mathbb{F}_2 \to \mathbb{F}_4 \ ; \ 1 \mapsto 2$$

for $k \in \mathbb{N}_{\geq 1}$.

3. We apply the same idea as in the previous questions but this time with functor $\mathbb{F}_2 \otimes_{\mathbb{Z}C_2} \_$ and then $\mathbb{F}_2 \otimes_{\mathbb{Z}C_2} G_\bullet \to \mathbb{F}_2 \otimes_{\mathbb{Z}C_4} G_\bullet$ to the bottom row :

$$\begin{tikzcd}
... \arrow[r] & \mathbb{F}_2 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{F}_2 \arrow[r, "0"] \arrow[d, "1"] & \mathbb{F}_2 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{F}_2 \arrow[d, "1"] \\
... \arrow[r] & \mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2
\end{tikzcd}$$

The multiplications by 2 and 4 become the 0 map since we are in $\mathbb{F}_2$. So we have that :

$$H_n(C_2;\mathbb{F}_2) \cong \mathbb{F}_2$$
$$H_n(C_4;\mathbb{F}_2) \cong \mathbb{F}_2$$

for all $n \in \mathbb{N}$. And then :

$$H_{2k}(f;\mathbb{F}_2) = id_{\mathbb{F}_2} : \mathbb{F}_2 \to \mathbb{F}_2$$
$$H_{2k+1}(f;\mathbb{F}_2) = 0 : \mathbb{F}_2 \to \mathbb{F}_2$$

for all $k \in \mathbb{N}$.

4. Notice that $\mathbb{Z}C_2$ is a $\mathbb{Z}C_4$-module induced by the projection $p : C_4 \to C_2$: For $x \in C_4, t \in C_2$ generators we have $x \cdot 1 := p(x) \cdot 1 = t \cdot 1 = t$ (with $1 \in \mathbb{Z}C_2$), and extend linearly. In a similar fashion as in point 1, we want to extend the identity on $\mathbb{Z}$ to a map $g : G_\bullet \to F_\bullet$, such that the following diagram commutes :

$$\begin{tikzcd}
... \arrow[r] & \mathbb{Z}C_4 \arrow[r, "t-1"] \arrow[d, "g_3"] & \mathbb{Z}C_4 \arrow[r, "t^3+t^2+t+1"] \arrow[d, "g_2"] & \mathbb{Z}C_4 \arrow[r, "t-1"] \arrow[d, "g_1"] & \mathbb{Z}C_4 \arrow[r, "\epsilon"] \arrow[d, "g_0"] & \mathbb{Z} \arrow[d, "id"] \\
... \arrow[r] & \mathbb{Z}C_2 \arrow[r, "t-1"] & \mathbb{Z}C_2 \arrow[r, "t+1"] & \mathbb{Z}C_2 \arrow[r, "t-1"] & \mathbb{Z}C_2 \arrow[r, "\epsilon"] & \mathbb{Z}
\end{tikzcd}$$

We can take $g_0 = p$ and we want to find $g_1 : \mathbb{Z}C_4 \to \mathbb{Z}C_2$ such that the following diagram commutes (once again, each g_i is uniquely determined by its image of 1) :

$$\begin{tikzcd}
\mathbb{Z}C_4 \arrow[r, "t-1"] \arrow[d, "g_1"] & \mathbb{Z}C_4 \arrow[d, "p"] \\
\mathbb{Z}C_2 \arrow[r, "t-1"] & \mathbb{Z}C_2
\end{tikzcd}$$

It should satisfy :

$$(t-1)g_1 = p(t-1) = (t-1) \text{ so we can take } g_1(1) = 1 \text{ i.e } g_1 = p$$

Moving on to the next map, we're looking for $g_2 : \mathbb{Z}C_4 \to \mathbb{Z}C_2$ making the following diagram commute :

$$\begin{tikzcd}
\mathbb{Z}C_4 \arrow[r, "t^3+t^2+t+1"] \arrow[d, "g_2"] & \mathbb{Z}C_4 \arrow[d, "p"] \\
\mathbb{Z}C_2 \arrow[r, "t+1"] & \mathbb{Z}C_2
\end{tikzcd}$$

We get :

$$(t+1)g_2 = p(t^3+t^2+t+1) = (t+1+t+1) = 2(t+1) \text{ so we can take } g_2 = 2p$$

Going on with this method we have that $g_3 = g_2 = 2p, g_5 = g_4 = 4p$ and we begin to see a pattern. Inductively we have that $g_{2n} = g_{2n+1} = 2^n p$ for every $n \in \mathbb{Z}$.

5. Applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}C_4} \_$ and then change the base ring of the bottom row's tensor products $\mathbb{Z} \otimes_{\mathbb{Z}C_4} F_\bullet \to \mathbb{Z} \otimes_{\mathbb{Z}C_2} F_\bullet$ on the whole diagram we obtain the following for the lowest numbers of the resolution :

$$\begin{tikzcd}
... \arrow[r] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "2"] & \mathbb{Z} \arrow[r, "4"] \arrow[d, "2"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "1"] & \mathbb{Z} \arrow[d, "1"] \\
... \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}
\end{tikzcd}$$

and at step $2n \in \mathbb{Z}$ of the resolution :

$$\begin{tikzcd}
... \arrow[r] & \mathbb{Z} \arrow[r, "4"] \arrow[d, "2^{n+1}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "2^n"] & \mathbb{Z} \arrow[r, "4"] \arrow[d, "2^n"] & \mathbb{Z} \arrow[r] \arrow[d, "2^{n-1}"] & ... \\
... \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r] & ...
\end{tikzcd}$$

Thus we obtain for every $n \in \mathbb{N}_{\geq 1}$:

$$H_0(g;\mathbb{Z}) = id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$$

$$H_1(g;\mathbb{Z}) = p : \mathbb{F}_4 \to \mathbb{F}_2$$

$$H_{2n}(g;\mathbb{Z}) = 0 : 0 \to 0$$

$$H_{2n+1}(g;\mathbb{Z}) = 0 : \mathbb{F}_4 \to \mathbb{F}_2$$

for the last one (H_{2n+1}), notice that the map $2^n : \mathbb{Z} \to \mathbb{Z}$ after taking the quotients gives us a map $[2^n] : \mathbb{F}_4 \to \mathbb{F}_2$ where we send $[1]$ mod 4 to $[2^n]$ mod 2 which is $\equiv 0$ for $n \geq 1$.

6. Similarly, applying the functor $\mathbb{F}_2 \otimes_{\mathbb{Z}C_4} \_$ to the diagram we constructed in point 4, and then $\mathbb{F}_2 \otimes_{\mathbb{Z}C_4} F_\bullet \to \mathbb{F}_2 \otimes_{\mathbb{Z}C_2} F_\bullet$ to the bottom row, we get the following diagram :

$$\begin{tikzcd}
... \arrow[r] & \mathbb{F}_2 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{F}_2 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{F}_2 \arrow[r, "0"] \arrow[d, "1"] & \mathbb{F}_2 \arrow[d, "1"] \\
... \arrow[r] & \mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2
\end{tikzcd}$$

with 0 maps for all the other maps of the diagram. Hence we get for every $n \in \mathbb{N}$:

$$H_0(g;\mathbb{F}_2) = H_1(g;\mathbb{F}_2) = id_{\mathbb{F}_2} : \mathbb{F}_2 \to \mathbb{F}_2$$

$$H_n(g;\mathbb{F}_2) = 0 : \mathbb{F}_2 \to \mathbb{F}_2$$

◊ **Exercice 2. Coefficients modules.** Let G be a group and $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ a short exact sequence of $\mathbb{Z}G$-modules.

1. Prove that there is a connecting homomorphism $\partial\colon H_{n+1}(G;M'') \to H_n(G;M')$ such that there is a long exact sequence in homology
$$\dots H_{n+1}(G;M'') \xrightarrow{\partial} H_n(G;M') \xrightarrow{i_*} H_n(G;M) \xrightarrow{p_*} H_n(G;M'') \xrightarrow{\partial} \dots$$
2. Prove the analogous statement in cohomology.
3. Compute $H_n(C_2;\mathbb{Z}/k)$ for any n and k (the coefficients have the trivial module structure).
4. Compute the connecting homomorphism associated to the exact sequence $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$ of trivial modules, in homology and cohomology (for the group C_2). This connecting homomorphism is called the *Bockstein homomorphism.*

Solution 2. (by Douglas and Fabien)

1. Let $F_\bullet$ be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. In particular, for each $n \in \mathbb{N}$, the module F_n is flat and we have a short exact sequence
$$0 \to F_n \otimes_G M' \to F_n \otimes_G M \to F_n \otimes_G M'' \to 0.$$
Pasting these together yields a short exact sequence
$$0 \to F_\bullet \otimes_G M' \to F_\bullet \otimes_G M \to F_\bullet \otimes_G M'' \to 0.$$

of chain complexes. Indeed it is immediate to see that following diagram commutes :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
\cdots \arrow[r] & F_n \otimes_{\mathbb{Z}G} M' \arrow[r, "\delta_n \otimes \mathrm{id}_{M'}"'] \arrow[d, "\mathrm{id}_{F_n} \otimes i"'] & F_{n-1} \otimes_{\mathbb{Z}G} M' \arrow[r] \arrow[d, "\mathrm{id}_{F_{n-1}} \otimes i"'] & \cdots \arrow[r] & F_0 \otimes_{\mathbb{Z}G} M' \arrow[d, "\mathrm{id}_{F_0} \otimes i"'] \\
\cdots \arrow[r] & F_n \otimes_{\mathbb{Z}G} M \arrow[r, "\delta_n \otimes \mathrm{id}_{M}"'] \arrow[d, "\mathrm{id}_{F_n} \otimes p"'] & F_{n-1} \otimes_{\mathbb{Z}G} M \arrow[r] \arrow[d, "\mathrm{id}_{F_{n-1}} \otimes p"'] & \cdots \arrow[r] & F_0 \otimes_{\mathbb{Z}G} M \arrow[d, "\mathrm{id}_{F_0} \otimes p"'] \\
\cdots \arrow[r] & F_n \otimes_{\mathbb{Z}G} M'' \arrow[r, "\delta_n \otimes \mathrm{id}_{M''}"'] \arrow[d] & F_{n-1} \otimes_{\mathbb{Z}G} M'' \arrow[r] \arrow[d] & \cdots \arrow[r] & F_0 \otimes_{\mathbb{Z}G} M'' \arrow[d] \\
 & 0 & 0 & & 0
\end{tikzcd},$$

where $\{\delta_n\}_{n\in\mathbb{N}}$ are the differential of $F_\bullet$.

Let us denote by $\{d_n\}_{n\in\mathbb{N}}$, $\{d'_n\}_{n\in\mathbb{N}}$ and $\{d''_n\}_{n\in\mathbb{N}}$ the differentials of the chain complexes $F_\bullet \otimes_G M$, $F_\bullet \otimes_G M'$ and $F_\bullet \otimes_G M''$ respectively. Now fix some $n \in \mathbb{N}$ and consider the commutative diagram

$$\begin{tikzcd}
 & F_n \otimes_G M' / \operatorname{Im}(d'_{n+1}) \arrow[r] \arrow[d, "\overline{d'_n}"] & F_n \otimes_G M / \operatorname{Im}(d_{n+1}) \arrow[r] \arrow[d, "\overline{d_n}"] & F_n \otimes_G M'' / \operatorname{Im}(d''_{n+1}) \arrow[r] \arrow[d, "\overline{d''_n}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(d'_{n-1}) \arrow[r] & \operatorname{Ker}(d_{n-1}) \arrow[r] & \operatorname{Ker}(d''_{n-1}) &
\end{tikzcd},$$

where the vertical maps are the maps induced by the differentials and the horizontal maps are induced by the maps i and p. As the rows are exact (this can be verified by hand) we may apply the snake lemma to get an exact sequence

$$\operatorname{Ker}(\overline{d'_n}) \to \operatorname{Ker}(\overline{d_n}) \to \operatorname{Ker}(\overline{d''_n}) \to \operatorname{Coker}(\overline{d'_n}) \to \operatorname{Coker}(\overline{d_n}) \to \operatorname{Coker}(\overline{d''_n}).$$

Note that by definition we have

$$\operatorname{Ker}(\overline{d_n}) = H_n(F_\bullet \otimes_G M) = H_n(G; M)$$

and

$$\operatorname{Coker}(\overline{d_n}) = H_{n-1}(F_\bullet \otimes_G M) = H_{n-1}(G; M)$$

and similarly for the other chain complexes. Thus we get an exact sequence

$$H_n(G; M') \to H_n(G; M) \to H_n(G; M'') \to H_{n-1}(G; M') \to H_{n-1}(G; M) \to H_{n-1}(G; M'').$$

As n was arbitrary, we may glue all these exact sequences to get the desired result.

2. Let us now prove the analogous statement for cohomology. That is, there are homomorphisms

$$\partial : H^n(G; M'') \to H^{n+1}(G; M')$$

such that there is a long exact sequence

$$\cdots \to H^n(G; M') \xrightarrow{i_*} H^n(G; M) \xrightarrow{p_*} H^n(G; M'') \xrightarrow{\partial} H^{n+1}(G; M') \to \cdots .$$

Let $F_\bullet$ be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Now for each $n \in \mathbb{N}$, the functor $\operatorname{Hom}_{\mathbb{Z}G}(F_n, -)$ is exact as F_n is free and in particular projective. Indeed, the functor $\operatorname{Hom}_{\mathbb{Z}G}(F_n, -)$ is always left exact and the property that it maps every surjection to a surjection is exactly what it means for F_n to be projective.

This imply that we get as before a short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M'') \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M') \to 0$$

of cochain complexes as follows :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
\cdots & \mathrm{Hom}_{\mathbb{Z}G}(F_n, M') \arrow[l] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}G}(F_n,i)"] & \mathrm{Hom}_{\mathbb{Z}G}(F_{n-1}, M') \arrow[l, "\mathrm{Hom}_{\mathbb{Z}G}(\delta_n,M')"'] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}G}(F_{n-1},i)"] & \cdots \arrow[l] & \mathrm{Hom}_{\mathbb{Z}G}(F_0, M') \arrow[l] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}G}(F_0,i)"] \\
\cdots & \mathrm{Hom}_{\mathbb{Z}G}(F_n, M) \arrow[l] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}G}(F_n,p)"] & \mathrm{Hom}_{\mathbb{Z}G}(F_{n-1}, M) \arrow[l, "\mathrm{Hom}_{\mathbb{Z}G}(\delta_n,M')"'] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}G}(F_{n-1},p)"] & \cdots \arrow[l] & \mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \arrow[l] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}G}(F_0,p)"] \\
\cdots & \mathrm{Hom}_{\mathbb{Z}G}(F_n, M'') \arrow[l] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}G}(F_{n-1}, M'') \arrow[l, "\mathrm{Hom}_{\mathbb{Z}G}(\delta_n,M')"'] \arrow[d] & \cdots \arrow[l] & \mathrm{Hom}_{\mathbb{Z}G}(F_0, M'') \arrow[l] \arrow[d] \\
 & 0 & 0 & & 0
\end{tikzcd}$$

Now the exact same argument as in point 1. shows the claim. Notice that the order of arrows is now reversed as $\mathrm{Hom}_{\mathbb{Z}G}(F_n, -)$ is contravariant.

3. Let us show that

$$H_n(C_2, \mathbb{Z}/k\mathbb{Z}) = \begin{cases} \mathbb{Z}/k\mathbb{Z} \text{ if } n = 0 \\ C_2 \text{ if } 2|k \text{ and } n > 0 \\ 0 \text{ if } 2 \not| k \text{ and } n > 0 \end{cases}.$$

The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/k\mathbb{Z} \to 0$$

where i is the multiplication by k induces a long exact sequence

$$\cdots \to H_{n+1}(C_2, \mathbb{Z}/k\mathbb{Z}) \to H_n(C_2, \mathbb{Z}) \xrightarrow{i_*} H_n(C_2, \mathbb{Z}) \xrightarrow{p_*} H_n(C_2, \mathbb{Z}/k\mathbb{Z}) \to \cdots.$$

We recall that

$$H_n(C_2, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ C_2 \text{ if } n \geq 1 \text{ is odd} \\ 0 \text{ if } n \geq 1 \text{ is even} \end{cases}.$$

Hence it is immediate that

$$H_0(C_2, \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/\operatorname{Ker}(p_*) = \mathbb{Z}/\operatorname{Im}(i_*) = \mathbb{Z}/k\mathbb{Z}.$$

Now let $n > 1$ be even. There is an exact sequence

$$0 \to H_n(C_2, \mathbb{Z}/k\mathbb{Z}) \to H_{n-1}(C_2, \mathbb{Z}) \cong C_2 \xrightarrow{i_*} H_{n-1}(C_2, \mathbb{Z}) \cong C_2,$$

where i_* is the multiplication by k. We deduce that

$$H_n(C_2, \mathbb{Z}/k\mathbb{Z}) \cong \operatorname{Ker}(i_*) = \begin{cases} C_2 \text{ if } 2|k \\ 0 \text{ if } 2 \not| k \end{cases}.$$

If $n > 1$ is odd the exact sequence

$$H_n(C_2, \mathbb{Z}) \cong C_2 \xrightarrow{i_*} H_n(C_2, \mathbb{Z}) \cong C_2 \xrightarrow{p_*} H_n(C_2, \mathbb{Z}/k\mathbb{Z}) \to 0$$

shows similarly that

$$H_n(C_2, \mathbb{Z}/k\mathbb{Z}) \cong C_2/\operatorname{Ker}(p_*) = C_2/\operatorname{Im}(i_*) = \begin{cases} C_2 \text{ if } 2|k \\ 0 \text{ if } 2 \nmid k \end{cases}$$

because i_* is the zero map whenever $2|k$ and it is the identity otherwise.
The only case remaining is $n = 1$. We have an exact sequence

$$C_2 \xrightarrow{i_*} C_2 \xrightarrow{p_*} H_1(C_2, \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to 0.$$

Notice that $\partial = 0$ as multiplication by k is an injection. This imply that p_* is a surjection by exactness. Hence,

$$H_1(C_2, \mathbb{Z}/k\mathbb{Z}) \cong C_2/\operatorname{Ker}(p_*) = C_2/\operatorname{Im}(i_*).$$

If 2 divides k, i_* is the zero map. If 2 does not divides k, the map i_* is the identity. This concludes the proof.

4. We start with the homology case.

Let us denote by $\partial_n : H_{n+1}(C_2, \mathbb{Z}/2\mathbb{Z}) \to H_n(C_2, \mathbb{Z}/2\mathbb{Z})$ the n-th connecting morphism. By questions 1. and 3. there is a long exact sequence

$$\cdots \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\partial_n} \mathbb{Z}/2\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{p_*} \mathbb{Z}/2\mathbb{Z} \to \cdots .$$

We have that $\partial_n : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is either the zero map or the identity. We will compute the induced maps $i_* : H_n(C_2, \mathbb{Z}/2\mathbb{Z}) \to H_n(C_2, \mathbb{Z}/4\mathbb{Z})$ at each degree to deduce ∂_n. Let us take the periodic free resolution $F_\bullet$ of $\mathbb{Z}$ as in Example 4.5 of the lecture notes, where each F_n is a copy of $\mathbb{Z}C_2$. There is a commutating diagram

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \mathbb{Z}/2\mathbb{Z} \arrow[d, "i"] \\
\cdots \arrow[r, "2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/4\mathbb{Z}
\end{tikzcd},$$

where the first row is $F_\bullet \otimes_{\mathbb{Z}C_2} \mathbb{Z}/2\mathbb{Z}$, the second row is $F_\bullet \otimes_{\mathbb{Z}C_2} \mathbb{Z}/4\mathbb{Z}$ and the vertical maps are the maps induced by i on tensor products. The maps i_* are induced by these vertical morphisms on homology groups. Hence we deduce that i_* is is the injection $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ on degree 0, the zero map on odd degrees $n \geq 1$ and the identity on even degrees $n \geq 1$. By the above long exact sequence this implies that

$$\partial_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} \text{ if n is odd} \end{cases} .$$

We now do the cohomology case.

By Question 2 we have a long exact sequence

$$\cdots \to H^n(C_2; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i_*} H^n(C_2; \mathbb{Z}/4\mathbb{Z}) \xrightarrow{p_*} H^n(C_2; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial_n} H^{n+1}(C_2; \mathbb{Z}/2\mathbb{Z}) \to \cdots .$$

Let $F_\bullet$ be the same resolution as above, as in Example 4.5 of the lecture notes. From this resolution, we can construct the two following cochain complexes :

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "i_0 = i\circ -"] & \operatorname{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "i_1 = i\circ -"] & \operatorname{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "i_2 = i\circ -"] & \cdots \\
\operatorname{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, \mathbb{Z}/4\mathbb{Z}) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, \mathbb{Z}/4\mathbb{Z}) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, \mathbb{Z}/4\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd},$$

where the first row is $\mathrm{Hom}_{\mathbb{Z}C_2}(F_\bullet, \mathbb{Z}/2\mathbb{Z})$, the second row is $\mathrm{Hom}_{\mathbb{Z}C_2}(F_\bullet, \mathbb{Z}/4\mathbb{Z})$ and the vertical maps are the map induced by i.
As $\mathrm{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, M) \cong M$ for any module M, this is equivalent to the diagram

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & \cdots \\
\mathbb{Z}/4\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "0"] & \cdots
\end{tikzcd},$$

where the horizontal maps come from the free periodic resolution of $\mathbb{Z}$. By definition,

$$H^n(C_2; \mathbb{Z}/k\mathbb{Z}) = H^n(\mathrm{Hom}_{\mathbb{Z}C_2}(F_\bullet; \mathbb{Z}/k\mathbb{Z}))$$

so we get

$$H^n(C_2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

and

$$H^n(C_2; \mathbb{Z}/4\mathbb{Z}) = \begin{cases} \mathbb{Z}/4\mathbb{Z} \text{ if } n = 0 \\ \mathbb{Z}/2\mathbb{Z} \text{ if } n \geq 1 \end{cases}.$$

Thus our long exact sequence becomes

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}/4\mathbb{Z} \xrightarrow{p_*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\partial_0} \cdots \xrightarrow{\partial_n} \mathbb{Z}/2\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{p_*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\partial_{n+1}} \cdots$$

Moreover, the vertical maps i of the above diagram induce the maps i_* on cohomology groups. On degree 0, we get that i_* is the injection $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$. On odd degree $n \geq 1$, it is the identity map and on even degree $n \geq 1$, it is the zero map.
Using this in our long exact sequence, we get that

$$\partial_n = \begin{cases} \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}.$$

♢ **Exercice 3. Ext and extensions. Reminder from rings and modules ?** Let R be a ring, and A, B two left R-modules. Let ξ denote an extension of A by B in R-modules, i.e. a short exact sequence $0 \to B \to E \to A \to 0$. We denote by $\Theta(\xi) \in \mathrm{Ext}^1_R(A, B)$ the image of the identity under the connecting homomorphism $\partial\colon \mathrm{Hom}_R(A, A) \to \mathrm{Ext}^1_R(A, B)$ induced by the short exact sequence ξ (in the long "Hom-Ext" exact sequence). The objective of this exercise is to prove that Θ is a bijection between the set of equivalences classes of extensions of A by B and $\mathrm{Ext}^1_R(A, B)$. We also fix a projective cover $P \to A$ and call K its kernel.

1. Prove that $\Theta(\xi) = 0$ if and only if ξ splits (as the trivial extension).
2. Show that any element $x \in \mathrm{Ext}^1_R(A, B)$ is in the image of the connecting homomorphism ∂ induced by the short exact sequence $K \xrightarrow{j} P \to A$ (it is $\partial(\beta)$ for a homomorphism $\beta\colon K \to B$).
3. Let X be the pushout of j and β in R-modules, given by the cokernel of $(j, -\beta)$. Show that there is an extension $0 \to B \to X \to A \to 0$ and prove that Θ sends it to x.
4. Show that another preimage β' of x under ∂, as in 2, gives rise to an equivalent extension $0 \to B \to X' \to A \to 0$.
5. For any extension ξ, show that there is a map of extensions from P to E, which yields an isomorphism between the pushout of the left square and E.
6. Conclude that Θ is injective.

Solution 3. (by Raphaël and Xie)

1. Let

$$\xi : \quad 0 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 0$$

be a short exact sequence of left R-modules and consider the long exact sequence given to us by applying $\operatorname{Hom}_R(A, \cdot)$ to ξ :

$$0 \longrightarrow \operatorname{Hom}_R(A,B) \xrightarrow{i_*} \operatorname{Hom}_R(A,E) \xrightarrow{p_*} \operatorname{Hom}_R(A,A) \xrightarrow{\partial} \operatorname{Ext}^1_R(A,B) \xrightarrow{i_*} \operatorname{Ext}^1_R(A,E) \longrightarrow \cdots$$

with $i_*(f) = i \circ f, p_*(g) = p \circ g$ and ∂ the connecting morphism induced by ξ that makes the sequence exact.

If ξ splits, there exists $s : A \to E$ such that $p \circ s = \mathrm{id}_A$, so $\mathrm{id}_A \in \operatorname{Im}(q_*)$, hence by exactness of the sequence above $\partial(\mathrm{id}_A) = 0$.

Conversely, if $\partial(\mathrm{id}_A) = 0$, exactness of the sequence above gives $\mathrm{id}_A = p_*(s) = p \circ s$ for some $s : A \to E$, so ξ splits.

2. As stated, let $\pi : P \to A$ be a projective cover with kernel K :

$$\xi_P : \quad 0 \longrightarrow K \xrightarrow{j} P \xrightarrow{\pi} A \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(\cdot, B)$ yields the exact sequence

$$\ldots \to \operatorname{Hom}_R(P,B) \to \operatorname{Hom}_R(K,B) \xrightarrow{\partial} \operatorname{Ext}^1_R(A,B) \to \operatorname{Ext}^1_R(P,B) = 0 \to \ldots$$

with the last equality $\operatorname{Ext}^1_R(P,B) = 0$ a result of P being projective. In particular this means that $\partial : \operatorname{Hom}_R(K,B) \to \operatorname{Ext}^1_R(A,B)$ is surjective, i.e. every element of $\operatorname{Ext}^1_R(A,B)$ is $\partial(\beta)$ for some morphism $\beta : K \to B$.

3. Let $\beta : K \to B$ an R-modules morphism, and let X be the pushout of $j : K \to P$ and β, i.e. :

$$X = (B \oplus P)/\operatorname{Im}(j, -\beta).$$

The goal of the exercice is to show that there exists an extension ξ_β

$$\xi_\beta : \quad 0 \longrightarrow B \to X \to A \longrightarrow 0$$

and that $\Theta(\xi_\beta) := \partial(Id_A) = \partial_P(\beta)$, with ∂ the connecting morphism $\partial : \operatorname{Hom}_R(A,A) \to \operatorname{Ext}^1_R(A,B)$ induced by ξ_β, while ∂_P is the connecting morphism $\partial_P : \operatorname{Hom}_R(K,B) \to \operatorname{Ext}^1_R(A,B)$ induced by the short exact sequence $\xi_P : 0 \to K \to P \to A \to 0$.

Let $\iota : B \to X$ be induced by the inclusion $B \to B \oplus P$ and $\rho : X \to A$ be induced by the projection $B \oplus P \to P \xrightarrow{q} A$ (note that ρ is well defined on X as $q \circ j = 0$). We show that

$$0 \longrightarrow B \xrightarrow{\iota} X \xrightarrow{\rho} A \longrightarrow 0$$

is exact. It is obvious that ρ is surjective and that $\operatorname{Im}(\iota) \subset \operatorname{Ker}(\rho)$. We need to show that ι is injective and that $\operatorname{Ker}(\rho) \subset \operatorname{Im}(\iota)$.

First inectivity of ι : let $b \in B$ with $\iota(b) = 0$, then by construction of X this means that there exists $k \in K$ s.t. $b = \beta(k)$ and $j(k) = 0$. Injectivity of j implies that $k = 0$ therefore $b = 0$, meaning that ι is injective.

For the inclusion $\operatorname{Ker}(\rho) \subset \operatorname{Im}(\iota)$: let $(b,p) \in B \oplus P$ s.t. $\overline{(b,p)} \in \operatorname{Ker}(\rho)$, then $q(p) = 0$, therefore $p = j(k)$ for some $k \in K$, but $\overline{(b, j(k))} = \overline{(b + \beta(k), 0)} = \iota(b + \beta(k))$ i.e. $\overline{(b,p)} \in \operatorname{Im}(\iota)$, meaning we have $\operatorname{Ker}(\rho) \subset \operatorname{Im}(\iota)$.

By construction of the pushout X, $j(k) = \beta(k)$ in X for all $k \in K$, meaning the following diagram commutes :

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[r, "\pi"] \arrow[d, "\iota_P"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "\iota"] & X \arrow[r, "\rho"] & A \arrow[r] & 0
\end{tikzcd}$$

with ι_P the map induced by the inclusion $P \subset B \oplus P$. In particular, ξ_β is the pushout extension of ξ_P by β.

Let us now remember the following characterizations of the connecting morphisms ∂ and ∂_P. In particular, $\partial(\mathrm{id}_A)$ is the class of the pullback extension of ξ_β obtained from id_A (i.e. ξ_β), while $\partial_P(\beta)$ is the class of the pushout extension of ξ_P obtained from β (also ξ_β). Both are the same, hence our result.

4. Let $\beta, \beta' \in \mathrm{Hom}_R(K, B)$ represent the same class $x \in \mathrm{Ext}^1_R(A, B)$ via the connecting map $\partial : \mathrm{Hom}_R(K, B) \to \mathrm{Ext}^1_R(A, B)$. Then $\beta' - \beta$ lies in the image of $j^* : \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(K, B)$, so there exists $\varphi \in \mathrm{Hom}_R(P, B)$ with
$$\beta' = \beta + \varphi \circ j.$$

   Form the cokernels
$$X = (B \oplus P)/\operatorname{im}(\beta, -j), \qquad X' = (B \oplus P)/\operatorname{im}(\beta', -j).$$

   Define an R-linear map $T : B \oplus P \to B \oplus P$ by
$$T(b, p) = (b + \varphi(p),\, p).$$

   Then T is an automorphism with inverse $T^{-1}(b, p) = (b - \varphi(p), p)$, and a straightforward computation gives for every $k \in K$:
$$T\big(\beta(k), -j(k)\big) = \big(\beta(k) + \varphi(j(k)), -j(k)\big) = \big(\beta'(k), -j(k)\big).$$

   Hence T maps $\operatorname{im}(\beta, -j)$ isomorphically onto $\operatorname{im}(\beta', -j)$, so it induces an isomorphism $\overline{T} : X \xrightarrow{\sim} X'$ of cokernels.
   Finally, $\overline{T}$ fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\overline{T}"', "\wr"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

   because on representatives $q(b, 0) \mapsto q'(b, 0)$ (where q, q' are the quotient maps) and $\overline{T}$ preserves the projection to A (it fixes the P-coordinate). Thus the two short exact sequences are equivalent.

5. Consider
$$\xi : \qquad 0 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 0, \qquad 0 \longrightarrow K \xrightarrow{j} P \xrightarrow{\pi} A \longrightarrow 0$$
   with P a projective cover of A.

   Since P is projective, there exists a lifting $s : P \to E$ with $p \circ s = \pi$. For $k \in K$ we have $\pi(k) = 0$, hence $p(s(k)) = 0$ and $s(k) \in i(B)$. Thus there is a unique $\beta : K \to B$ with $s|_K = i \circ \beta$.
   Define $\Phi : B \oplus P \to E$ by $\Phi(b, p) := i(b) + s(p)$. For $k \in K$,
$$\Phi\big(\beta(k), -j(k)\big) = i(\beta(k)) + s(-j(k)) = s(k) - s(j(k)) = 0,$$

so $\operatorname{im}(\beta,-j)\subset \operatorname{Ker}\Phi$. Therefore Φ factors through the cokernel

$$X=(B\oplus P)/\operatorname{im}(\beta,-j)$$

and we obtain $\overline{\Phi}: X\to E$.

Surjectivity : For any $e\in E$ let $a:=p(e)\in A$. Choose $p_0\in P$ with $\pi(p_0)=a$. Then $e-s(p_0)\in \operatorname{Ker} p=i(B)$, so $e=\Phi(b,p_0)$ for some $b\in B$; hence $\overline{\Phi}$ is surjective.

Injectivity : If $\overline{\Phi}(q(b,p))=0$ then $i(b)+s(p)=0$, so $\pi(p)=p(i(b)+s(p))=0$, hence $p\in K$. But then $s(p)=i(\beta(p))$, thus $i(b)+i(\beta(p))=0$, i.e. $b=-\beta(p)$. Therefore $(b,p)\in\operatorname{im}(\beta,-j)$, so $q(b,p)=0$. Hence $\overline{\Phi}$ is injective.

So $\overline{\Phi}: X\xrightarrow{\sim} E$ is an isomorphism of R-modules. One checks easily that $\overline{\Phi}$ identifies the inclusion $B\to X$ with $i: B\to E$ and commutes with the projections to A. Thus the pushout extension $0\to B\to X\to A\to 0$ is isomorphic (as an extension) to the given ξ.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\overline{\Phi}"', "\wr"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

6. Injectivity of Θ : Suppose two extensions

$$\xi_1:\ 0\to B\to E_1\to A\to 0,\qquad \xi_2:\ 0\to B\to E_2\to A\to 0$$

have the same image under Θ, i.e. $\Theta(\xi_1)=\Theta(\xi_2)\in \operatorname{Ext}^1_R(A,B)$.

Choose the projective presentation

$$0\longrightarrow K\xrightarrow{j} P\xrightarrow{\pi} A\longrightarrow 0,$$

and let $\beta_1,\beta_2\in\operatorname{Hom}_R(K,B)$ correspond to ξ_1,ξ_2 via the construction above. Equality of $\Theta(\xi_1)$ and $\Theta(\xi_2)$ means

$$\partial(\beta_1)=\partial(\beta_2),$$

where $\partial:\operatorname{Hom}_R(K,B)\to\operatorname{Ext}^1_R(A,B)$ is the connecting map. Hence $\beta_1-\beta_2$ lies in the image of $j^*:\operatorname{Hom}_R(P,B)\to\operatorname{Hom}_R(K,B)$, so there exists $\varphi: P\to B$ such that

$$\beta_2=\beta_1+\varphi\circ j.$$

By the previous result, β_1 and β_2 then yield equivalent extensions

$$0\to B\to X_1\to A\to 0\quad\text{and}\quad 0\to B\to X_2\to A\to 0.$$

But each X_i is isomorphic to E_i (by the canonical isomorphism constructed from P to E_i). Therefore ξ_1 and ξ_2 are equivalent as extensions.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, "\simeq"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Hence Θ is injective.

◊ **Exercice 4. Functorial properties of extensions.** Let $0\to A\to E\to G\to 1$ be an extension of G by an abelian group A. We fix homomorphisms $\alpha\colon G'\to G$ and $f\colon A\to A'$ a homomorphism of $\mathbb{Z}G$-modules.

1. Show that the pullback of extensions along α corresponds to $\alpha_*\colon H^2(G;A) \to H^2(G';\alpha^*(A))$.
2. Show that E acts on A' via the projection $E \to G$, so we can form the semi-direct product $A' \rtimes E$.
3. Let N be the subgroup in $A' \rtimes E$ generated by elements of the form $(-f(a), a)$. Show that N is a normal
4. Show that A' is a subgroup of $(A' \rtimes E)/N$ whose quotient is isomorphic to G.
5. Prove that the induced map of extensions corresponds to the map $f_*\colon H^2(G;A) \to H^2(G;A')$.

Solution 4. (Adapted from Qian and Zichun)
We write $0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$

1. Define the pullback along α by

$$E' = \{(g', e) \in G' \times E \mid \alpha(g') = \pi(e)\}.$$

Then

$$0 \longrightarrow A \xrightarrow{\iota'} E' \longrightarrow G' \longrightarrow 1$$

is an extension of G' by the G'-module α^*A, where the G'–action on A is given by $g' \cdot a = \alpha(g') \cdot a$. On the other hand, the pullback on cohomology is defined as :

$$\alpha_*\colon H^2(G;A) \longrightarrow H^2(G';\alpha^*A), \qquad [c] \mapsto [\,c \circ (\alpha \times \alpha)\,].$$

It suffices to show that for a set-theoretical section s of π and factor set $f\colon G \times G \to A$ such that $s(g)s(h) = \iota(f(g,h))s(gh)$, $f \circ (\alpha \times \alpha)$ is a factor set $G' \times G' \to A$ corresponding to a section s' of E'. Define $s' : G' \to E'$ by $s'(g) = (s(\alpha(g)), g)$.
We compute :

$$\begin{aligned} s'(g)s'(h) &= (s(\alpha(g)), g)(s(\alpha(h)), h) \\ &= (s(\alpha(g))s(\alpha(h)), gh) \\ &= (f(\alpha(g), \alpha(h))s(gh), gh) \\ &= (f(\alpha(g), \alpha(h)), 1)(s(\alpha(gh)), gh) \\ &= \iota'(f(\alpha(g), \alpha(h))s'(g'h') \end{aligned}$$

2. Since A' is a $\mathbb{Z}G$-module, each $g \in G$ acts on A'. For $e \in E$ define

$$e \cdot a' := \pi(e) \cdot a', \qquad (a' \in A').$$

This makes A' into an E-module, and we can form the semidirect product

$$A' \rtimes E = \{(a', e) \mid a' \in A',\ e \in E\}$$

with multiplication law

$$(a'_1, e_1) \cdot (a'_2, e_2) = \big(a'_1 + \pi(e_1) \cdot a'_2,\ e_1e_2\big).$$

3. Let N be the subgroup of $A' \rtimes E$ generated by all elements of the form $(-f(a), a), \quad a \in A$.
For $(x, e) \in A' \rtimes E$, let $g = \pi(e) \in G$. Recall

$$(x, e)^{-1} = \big(-g^{-1} \cdot x,\ e^{-1}\big).$$

Now compute the conjugation of a generator of N by (x, e) :

$$(x, e) \cdot (-f(a), a) \cdot (x, e)^{-1}.$$

First,

$$(x,e)\cdot(-f(a),a) = \big(x + g\cdot(-f(a)),\ ea\big).$$

Multiplying by $(x,e)^{-1} = (-g^{-1}\cdot x, e^{-1})$ gives

$$\begin{aligned}&(x + g\cdot(-f(a)),\ ea)\cdot(-g^{-1}\cdot x, e^{-1})\\ &\quad = \big(x + g\cdot(-f(a)) + \pi(ea)\cdot(-g^{-1}\cdot x),\ eae^{-1}\big).\end{aligned}$$

Since $\pi(ea) = \pi(e) = g$ (because $a \in A = \operatorname{Ker}\pi$), we have

$$\pi(ea)\cdot(-g^{-1}\cdot x) = g\cdot(-g^{-1}\cdot x) = -x.$$

Hence the A'–component simplifies to

$$x + g\cdot(-f(a)) - x = g\cdot(-f(a)) = -\,g\cdot f(a).$$

Because f is $\mathbb{Z}G$–linear, we have $g\cdot f(a) = f(g\cdot a)$. The E–component is eae^{-1}, which equals $g\cdot a$ since the conjugation action of E on A corresponds to the G–action on A.

Therefore,

$$(x,e)\,(-f(a),a)\,(x,e)^{-1} = \big(-\,f(g\cdot a),\ g\cdot a\big),$$

which is again an element of the form $(-f(a'),a')$ with $a' = g\cdot a \in A$. Thus every conjugate of a generator of N lies in N, and so N is a normal subgroup of $A' \rtimes E$.

4. First, we show that A' is a subgroup of $(A'\rtimes E)/N$. For that it suffices to show that the composition

   $$A' \to A'\rtimes E \to (A'\rtimes E)/N$$

   is injective. Take an element $a' \in A'$ such that its image under the above is $[0] \in (A'\rtimes E)/N$. Then, the image of a' in $A'\rtimes E$ is in N, so $(a',0)$ has the form of $(-f(a),a)$, but this gives $a' = 0$.

   Next, we show that A' is a normal subgroup. Indeed $(b',e)\cdot(a',0)\cdot(b',e)^{-1} = (-g\cdot a',0)$. We can take the quotient by A'.

   Then, we show that the quotient of $(A'\rtimes E)/N$ by A' is isomorphic to G. Consider $p : A'\rtimes E \to G$ sending $e = (b',(c,g)) \in A'\rtimes E$ to g.

   This group homomorphism factor through $(A'\rtimes E)/N$ because $p((-f(a),a)) = p((-f(a),(a,1))) = 1$. We get an induced homomorphism $p' : (A'\rtimes E)/N \to G$. It factors through the quotient $((A'\rtimes E)/N)/A'$ because $p'([(a',(0,1))]) = p((a',(0,1))) = 1$.

   We have now a group homomorphism $p'' : ((A'\rtimes E)/N)/A' \to G$. Then it suffices to prove p'' is bijection by constructing its inverse $q : G \to ((A'\rtimes E)/N)/A'$ by sending $g \in G$ to $[(0,(0,g))]$. We proved that the quotient is isomorphic to G.

5. Choose a normalized section $s\colon G \to E$ with $s(g)s(h) = \iota(c(g,h))\,s(gh)$, where $c \in Z^2(G;A)$. In $(A'\rtimes E)/N$, the section $\tilde{s}(g) = (0,s(g))$ satisfies :

   $$\tilde{s}(g)\tilde{s}(h) = (0, s(g)s(h)) = (0,\ \iota(c(g,h))\,s(gh)) \equiv (f(c(g,h)),\ s(gh)) \mod N.$$

   Hence the 2–cocycle representing this new extension is $f\circ c \in Z^2(G;A')$. Consequently, the cohomology class of $(A'\rtimes E)/N$ is

   $$f_*([c]) \in H^2(G;A').$$